

On the spectral curve of the difference Lamé operator

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Abstract

We give two "complementary" descriptions of the curve Γ parametrizing double-Bloch solutions to the difference analogue of the Lamé equation. The curve depends on a positive integer number ℓ and two continuous parameters: the "lattice spacing" η and the modular parameter τ . Apart from being a covering of the elliptic curve with the modular parameter τ , Γ is a hyperelliptic curve of genus 2ℓ . We also point out connections between the spectral curve and representations of the Sklyanin algebra.

1 Introduction

Schrödinger operators with a periodic potential usually have infinitely many gaps in the spectrum. Exceptional cases, when there are only a finite number of gaps, are of particular interest for the theory of ordinary differential equations as well as for applications. Their study goes back to classical works of the last century. The renewed interest to the theory of finite-gap operators is due to their role in constructing quasi-periodic exact solutions to non-linear integrable equations.

Among known examples of the finite-gap operators, the most familiar one is the classical Lamé operator

$$\mathcal{L} = -\frac{d^2}{dx^2} + \ell(\ell+1)\wp(x), \quad (1.1)$$

where $\wp(x)$ is the Weierstrass \wp -function and ℓ is a parameter. The finite gap property of higher Lamé operators for integer values of ℓ was established in [1]. If ℓ is a positive integer, then there exists a differential operator of order $2\ell+1$ that commutes with \mathcal{L} , so the Lamé operator has exactly ℓ gaps in the spectrum. Such a remarkable spectral property is a signification of a hidden algebraic symmetry underlying the spectral problem.

In [2], a connection between the finite-gap integration theory of soliton equations and the representation theory of Sklyanin algebra [3] was found and the following difference analogue of the Lamé operator was proposed:

$$L = \frac{\theta_1(2x - 2\ell\eta)}{\theta_1(2x)} e^{\eta\partial_x} + \frac{\theta_1(2x + 2\ell\eta)}{\theta_1(2x)} e^{-\eta\partial_x}. \quad (1.2)$$

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Here $\theta_1(x) \equiv \theta_1(x|\tau)$ is the odd Jacobi θ -function, ℓ is a non-negative integer and $\eta \in \mathbf{C}$ is a parameter which is assumed to belong to the fundamental parallelogram with vertices $0, 1, \tau, 1+\tau$. The origin of the operator (1.2) is traced back to Sklyanin's paper [4] of 1983, where a functional realization of the Sklyanin algebra was found. Namely, L coincides with one of the four generators of the Sklyanin algebra in the functional realization. Therefore, the Sklyanin algebra provides a natural algebraic framework for analyzing the spectral properties of the operator L . (A different algebraic approach to the difference analogues of the Lamé operators was proposed in [5].) Nowdays, another face of this operator is probably more familiar: it is the hamiltonian of the elliptic two-body Ruijsenaars model [6].

As is already expected from the relation to the Sklyanin algebra, the spectral problem $L\Psi = E\Psi$ is closely connected with the simplest one-site XYZ spin chain of spin ℓ at the site. Indeed, the operator L is proportional to the trace of the quantum L-operator of this model. Integrable spin chains of XYZ -type can be solved by the generalized algebraic Bethe ansatz [7],[8],[9]. In our case the Bethe ansatz approach amounts to looking for the eigenfunctions of the form

$$\Psi(x) = K^{x/\eta} \prod_{j=1}^{\ell} \theta_1(2x - 2x_j),$$

where K and x_j are parameters. If they are constrained by the system of Bethe equations

$$K^2 \frac{\theta_1(2x_i - 2\ell\eta)}{\theta_1(2x_i + 2\ell\eta)} = \prod_{j=1, j \neq i}^{\ell} \frac{\theta_1(2x_i - 2x_j - 2\eta)}{\theta_1(2x_i - 2x_j + 2\eta)}, \quad i = 1, \dots, \ell,$$

then Ψ is an eigenfunction of L . The energy E obtained from the eigenvalue equation at a particular value of x (say, $x = \ell\eta$),

$$E = K^{-1} \frac{\theta_1(4\ell\eta)}{\theta_1(2\ell\eta)} \prod_{j=1}^{\ell} \frac{\theta_1(2(\ell-1)\eta - 2x_j)}{\theta_1(2\ell\eta - 2x_j)},$$

is then a multivalued function of K . It becomes single-valued on the *spectral curve* Γ of the operator L , which, therefore, carries all the information about its spectral properties. The points P of the curve are solutions $P = \{K, x_1, \dots, x_\ell\}$ to the Bethe system. However, the description of the curve provided by the Bethe equations is neither the most economic nor very informative one (at least for small values of ℓ).

Let $\Psi(x)$ be a function on which the operator (1.2) acts. Putting $\Psi_n = \Psi(n\eta + x_0)$, we assign to (1.2) the difference Schrödinger operator $L\Psi_n = A_n\Psi_{n+1} + B_n\Psi_{n-1}$ with quasiperiodic coefficients. The spectrum of a generic operator of this form has a structure of Cantor set type. If η is a rational number, $\eta = P/(2Q)$, this operator has Q -periodic coefficients. In general, Q -periodic difference Schrödinger operators have Q stable bands in the spectrum.

It was proved [2] that for integer ℓ the operator L is algebraically integrable and, therefore, is a difference analogue of the classical Lamé operator which can be obtained from L in the limit $\eta \rightarrow 0$. Algebraic integrability of L implies, in particular, some extremely unusual spectral properties of this operator. Namely, the operator L given by eq. (1.2) for positive integer values of ℓ and arbitrary generic η has $2\ell+1$ stable bands (and

2ℓ gaps) in the spectrum. Its Bloch functions are parametrized by points $P = (w, E)$ of a hyperelliptic curve of genus 2ℓ defined by the equation

$$w^2 = c \prod_{i=1}^{2\ell+1} (E^2 - E_i^2), \quad (1.3)$$

where c is a constant (introduced here for consistency with the definition of w given below). Moreover, eigenfunctions of the operator L at the edges of bands $E = \pm E_i$ span an invariant functional subspace for all generators of the Sklyanin algebra.

The finite-gap property of the operator L means that there exists a difference operator W of finite order such that it can not be represented as a polynomial function of L , and that commutes with L : $[L, W] = 0$. (This is the difference version of the Novikov equation for coefficients of the operators L and W .) It is well-known from the early days of finite-gap theory that the ring of operators commuting with the finite-gap operator is isomorphic to a ring of meromorphic functions on the corresponding spectral curve with poles at "infinite points". For difference operators this was proved in [10], [11]. Therefore, the ring of operators commuting with L is generated by L and an operator W such that

$$W^2 = c \prod_{i=1}^{2\ell+1} (L^2 - E_i^2). \quad (1.4)$$

The variable w in (1.3) is eigenvalue of the operator W , i.e., $W\Psi = w\Psi$, where Ψ is a common eigenfunction of L and W . The hyperelliptic curve (1.3) has two "infinite points" ∞_{\pm} , where the function E has first order poles. In [13], for any algebraic curve with two punctures, a special basis in the ring of meromorphic functions with poles at the punctures was introduced. Due to the isomorphism between the ring of meromorphic functions with poles at ∞_{\pm} and the ring of commuting operators, there exist operators commuting with L such that their eigenvalues on the common (Baker-Akhiezer) eigenfunction coincide with the basis functions. The explicit form of these operators and the operator W in the case when L is the difference Lamé operator (1.2) was found in [12]. Remarkably, this commuting family (parametrized by a complex parameter) coincides with the Baxter Q -operator for the one-site XYZ -model with spin ℓ .

This paper is devoted to a more detailed analysis of the spectral curves of the difference Lamé operators for arbitrary positive integer values of ℓ . Let us mention that spectral curves of the classical Lamé operator (1.1) and its Treibich-Verdier generalizations [14] for small values of ℓ were studied in [15]. Section 2 contains some algebraic preliminaries on quantum transfer matrices and the Q -operator for the simplest one-site XYZ -model with spin $\ell \in \mathbf{Z}_+$. This algebraic framework is very helpful since it allows one to represent the equation of the spectral curve in the most compact explicit form. The exposition in the first part of Section 3 follows that of the paper [2]. Namely, we construct double-Bloch solutions to the difference Lamé equation and obtain the spectral curve defined by two equations for three variables. One of these variables is the eigenvalue E , the other two parametrize the Bloch multipliers of the solution. Next, we show that E can be eliminated leaving us with one equation for two variables. In this realization, the fact that the spectral curve is a covering of the elliptic curve is transparent. However, the hyperelliptic property of the curve is implicit. In Section 4 we present the curve in the explicit hyperelliptic form. Finally, Section 5 contains a few results on the connection with representations of the Sklyanin algebra.

2 Quantum transfer matrices and the Q -operator for the one-site XYZ -model with spin ℓ

This section contains selected ingredients of the quantum inverse scattering approach to XYZ spin chains [8] specified to the case of the one-site "chain" with spin ℓ . These constructions turn out to be particularly useful for analyzing the spectral curve of the difference Lamé operator.

We begin with a few formulas related to the Sklyanin algebra and its representations. Definitions and transformation properties of the Jacobi θ -functions $\theta_a(x|\tau)$ are listed in Appendix A. For brevity, we write $\theta_a(x|\tau) \equiv \theta_a(x)$.

The elliptic quantum L-operator is the matrix

$$L(u) = \frac{1}{2} \begin{pmatrix} \theta_1(u)\mathcal{S}_0 + \theta_4(u)\mathcal{S}_3 & \theta_2(u)\mathcal{S}_1 + \theta_3(u)\mathcal{S}_2 \\ \theta_2(u)\mathcal{S}_1 - \theta_3(u)\mathcal{S}_2 & \theta_1(u)\mathcal{S}_0 - \theta_4(u)\mathcal{S}_3 \end{pmatrix} \quad (2.1)$$

with non-commutative matrix elements. Specifically, \mathcal{S}_a are difference operators in a complex variable x :

$$\mathcal{S}_a = \frac{\theta_{a+1}(2x - 2\ell\eta)}{\theta_1(2x)} e^{\eta\partial_x} - \frac{\theta_{a+1}(-2x - 2\ell\eta)}{\theta_1(2x)} e^{-\eta\partial_x}, \quad (2.2)$$

introduced by Sklyanin [4] in 1983. Comparing with (1.2), we identify $L = \mathcal{S}_0$. The four operators \mathcal{S}_a obey the commutation relations of the Sklyanin algebra¹:

$$\begin{aligned} (-1)^{\alpha+1} I_{\alpha 0} \mathcal{S}_\alpha \mathcal{S}_0 &= I_{\beta\gamma} \mathcal{S}_\beta \mathcal{S}_\gamma - I_{\gamma\beta} \mathcal{S}_\gamma \mathcal{S}_\beta, \\ (-1)^{\alpha+1} I_{\alpha 0} \mathcal{S}_0 \mathcal{S}_\alpha &= I_{\gamma\beta} \mathcal{S}_\beta \mathcal{S}_\gamma - I_{\beta\gamma} \mathcal{S}_\gamma \mathcal{S}_\beta \end{aligned} \quad (2.3)$$

with the structure constants $I_{ab} = \theta_{a+1}(0)\theta_{b+1}(2\eta)$. Here $a, b = 0, \dots, 3$ and $\{\alpha, \beta, \gamma\}$ stands for any cyclic permutation of $\{1, 2, 3\}$. The relations of the Sklyanin algebra are equivalent to the condition that the L-operator satisfies the "RLL = LLR" relation with the standard elliptic R -matrix.

The parameter ℓ in (2.2) is called spin of the representation. If necessary, we write $\mathcal{S}_a = \mathcal{S}_a^{(\ell)}$ to indicate the dependence on ℓ . When $\ell \in \frac{1}{2}\mathbf{Z}_+$, these operators have a finite-dimensional invariant subspace, namely, the space $\Theta_{4\ell}^+$ of even θ -functions of order 4ℓ (see Appendix A). This is the representation space of the $(2\ell + 1)$ -dimensional irreducible representation (of series a)) of the Sklyanin algebra.

Trace of $L(u)$ (in the two-dimensional auxiliary space), that is the simplest quantum transfer matrix $T_1(u)$, is proportional to $L = \mathcal{S}_0$:

$$T_1(u) = \text{tr}L(u) = \theta_1(u)\mathcal{S}_0. \quad (2.4)$$

The whole family of commuting transfer matrices $T_s(u)$, $s \in \mathbf{Z}_+$, is obtained from (2.1) via the fusion procedure [16]. We denote them by $T_s(u)$, $s \in \mathbf{Z}_+$. They commute for all values

¹The standard generators of the Sklyanin algebra [3] are related to ours as follows: $S_a = (i)^{\delta_{a,2}} \theta_{a+1}(\eta) \mathcal{S}_a$.

of s and u : $[T_s(u), T_{s'}(u')] = 0$. Here we do not need to recall the fusion procedure itself and refer the reader to [17] and [19], where integrable magnets of higher spin in the XXZ and XYZ case, respectively, were constructed by means of the fusion procedure. For our purpose it is enough to *define* the operators $T_s(u)$ by the recurrence relations (known as the fusion relations):

$$\begin{aligned} s < 2\ell : \quad & T_1(u-s\eta)T_s(u+\eta) = T_{s+1}(u) + \theta_1(u-s\eta-2\ell\eta)\theta_1(u-s\eta+2(\ell+1)\eta)T_{s-1}(u+2\eta), \\ s = 2\ell : \quad & T_1(u-2\ell\eta)T_{2\ell}(u+\eta) = \theta_1(u)T_{2\ell+1}(u) + \theta_1(u+2\eta)\theta_1(u-4\ell\eta)T_{2\ell-1}(u+2\eta), \\ s > 2\ell : \quad & T_1(u-s\eta)T_s(u+\eta) = \theta_1(u-s\eta+2\ell\eta)T_{s+1}(u) + \theta_1(u-s\eta-2\ell\eta)T_{s-1}(u+2\eta) \end{aligned} \quad (2.5)$$

with the "initial condition" $T_0(u) = \text{id}$, $T_1(u) = \theta_1(u)\mathcal{S}_0$. There is a useful determinant formula which solves the recurrence relations and represents $T_s(u)$ through $T_1(u)$ [18]:

$$\begin{aligned} T_s(u) &= \det(\mathbf{T}_{ij}(s, u))_{1 \leq i, j \leq s}, \quad 0 \leq s \leq 2\ell, \\ T_s(u) &= \left(\prod_{i=1}^{s-2\ell} \theta_1(u+(2\ell+2i-s-1)\eta) \right)^{-1} \det(\mathbf{T}_{ij}(s, u))_{1 \leq i, j \leq s}, \quad s > 2\ell, \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} \mathbf{T}_{ij}(s, u) &= \delta_{i,j-1}\theta_1(u+(s-2\ell-1-2i)\eta) + \delta_{i,j+1}\theta_1(u+(s+2\ell+3-2i)\eta) \\ &\quad + \delta_{i,j}T_1(u+(s+1-2i)\eta). \end{aligned} \quad (2.7)$$

Let Ψ be a common eigenfunction of $L = \mathcal{S}_0$ and $T_s(u)$, and let E be the eigenvalue of L : $L\Psi = E\Psi$. Then the eigenvalue of $T_s(u)$ is a polynomial in E of degree s , which we denote by $T_s(u, E)$:

$$\begin{cases} \mathcal{S}_0\Psi &= E\Psi, \\ T_s(u)\Psi &= T_s(u, E)\Psi. \end{cases} \quad (2.8)$$

The eigenvalues $T_s(u, E)$ are determined by eqs. (2.6), (2.7), where $T_1(u+(s+1-2i)\eta)$ is replaced by $E\theta_1(u+(s+1-2i)\eta)$. As a function of u , $T_s(u, E)$ for $1 \leq s \leq 2\ell$ is easily seen to belong to the space Θ_s of θ -functions of order s (for the precise definition see Appendix A). An important fact, not obvious from the definition (2.5), is that $T_s(u, E)$ for all $s > 2\ell$ belong to the space $\Theta_{2\ell}$, i.e., the denominator in (2.6) cancels. In particular,

$$\begin{aligned} T_{2\ell+1}(u, E) &= \frac{1}{\theta_1(u)} \det \left(\delta_{i,j} E\theta_1(u+2(\ell+1-i)\eta) + \right. \\ &\quad \left. + \delta_{i,j-1}\theta_1(u+2(2\ell+1-i)\eta) + \delta_{i,j+1}\theta_1(u-2(i-1)\eta) \right)_{1 \leq i, j \leq 2\ell+1} \end{aligned} \quad (2.9)$$

is holomorphic at $u = 0$.

The full family of operators commuting with \mathcal{S}_0 is generated by Baxter's Q -operator $\hat{Q}(u)$. Moreover, the operators $\hat{Q}(u)$ commute with all the transfer matrices and among themselves: $[T_s(u), \hat{Q}(u')] = 0$, $[\hat{Q}(u), \hat{Q}(u')] = 0$. They obey the famous Baxter T - Q -relation

$$\theta_1(u-2\ell\eta)\hat{Q}(u+2\eta) + \theta_1(u+2\ell\eta)\hat{Q}(u-2\eta) = T_1(u)\hat{Q}(u). \quad (2.10)$$

We also recall the formula for $T_s(u, E)$ through eigenvalues $Q(u)$ of the Q -operator [17], [19]:

$$\begin{aligned} T_s(u, E) &= \frac{Q(u + (s+1)\eta)Q(u - (s+1)\eta)}{\prod_{p=1}^{2\ell-s} \theta_1(u + (2\ell+1-s-2p)\eta)} \\ &\times \sum_{j=0}^s \frac{\prod_{q=1}^{2\ell} \theta_1(u + (2\ell+1+s-2j-2q)\eta)}{Q(u + (s-2j-1)\eta)Q(u + (s-2j+1)\eta)}, \quad 1 \leq s \leq 2\ell. \end{aligned} \quad (2.11)$$

If $s \geq 2\ell$, there is no denominator in the prefactor.

Let $\ell \in \mathbf{Z}_+$. In this case a commuting family, equivalent to the Q -operator, was explicitly constructed in [12]. Consider the operators

$$A_\lambda = \sum_{k=0}^\ell A_k(x, \lambda) e^{(2k\eta - \ell\eta + \lambda)\partial_x}, \quad (2.12)$$

where $\lambda \in \mathbf{C}$ and

$$\begin{aligned} A_k(x, \lambda) &= (-1)^k \frac{[\ell]!}{[2\ell]!} \left[\begin{array}{c} \ell \\ k \end{array} \right] \prod_{j=0}^{\ell-k-1} \frac{\theta_1(2x + 2(\ell-j)\eta)\theta_1(2\lambda + 2(\ell-j)\eta)}{\theta_1(2x + 2\lambda + 2(k-j)\eta)} \\ &\times \prod_{j=0}^{k-1} \frac{\theta_1(2x - 2(\ell-j)\eta)\theta_1(2\lambda - 2(\ell-j)\eta)}{\theta_1(2x + 2\lambda + 2(k+j-\ell)\eta)}. \end{aligned} \quad (2.13)$$

If $k = 0$ or $k = \ell$, the second (respectively, the first) product is absent. Here and below we use the "elliptic factorial" and "elliptic binomial" notation:

$$\begin{aligned} [n]! &= \prod_{j=1}^n [j], \quad [j] \equiv \theta_1(2j\eta), \\ \left[\begin{array}{c} n \\ m \end{array} \right] &\equiv \frac{[n]!}{[m]![n-m]!}. \end{aligned} \quad (2.14)$$

The main property of the operators (2.12) proved in [12] is their commutativity for all values of λ : $[A_\lambda, A_{\lambda'}] = 0$. For generic λ the chain of shifts in (2.12) starts from $-\ell\eta + \lambda$ and all the $\ell+1$ coefficients in (2.12) are non-zero. However, for $\lambda = l\eta$, $l = \ell, \ell-1, \dots, -\ell$ only $\ell - |l| + 1$ of them are non-zero. (For example, $A_{\ell\eta} = 1$, $A_{(\ell-1)\eta} = ([\ell]/[2\ell])\mathcal{S}_0$.)

By difference operator in the next theorem we mean a finite sum $\sum_k f_k(x) e^{k\eta\partial_x}$ with integer k . (So A_λ are difference operators, in this sense, only if $\lambda = m\eta$ with integer m .)

Theorem 2.1 [12] *The ring of difference operators commuting with $L = \mathcal{S}_0$ (1.2) is generated by L and $A \equiv A_{(\ell+1)\eta}$.*

It is convenient to introduce the following special notation: $A \equiv A_{(\ell+1)\eta}$, $\bar{A} \equiv A_{-(\ell+1)\eta}$, $W = A - \bar{A}$. The role of the operator W (the very one entering eq. (1.4)) for representations of the Sklyanin algebra is clarified in Sect. 5.

We conclude this section by listing some properties of the operators A_λ which will be useful in the sequel.

a) *The Baxter T-Q-relation.* It has been proved in [12] that the A_λ obey the following operator identity:

$$\mathcal{S}_0 A_\lambda = \frac{\theta_1(2\lambda - 2\ell\eta)}{\theta_1(2\lambda)} A_{\lambda+\eta} + \frac{\theta_1(2\lambda + 2\ell\eta)}{\theta_1(2\lambda)} A_{\lambda-\eta}, \quad (2.15)$$

which allows us to identify A_λ with the Q -operator: $A_\lambda = \hat{Q}(2\lambda)$ (cf. (2.10), (2.4)). In other words, A_λ can be regarded as an operator solution to the Baxter relation. The second operator solution to the second order equation (2.15) is $A_{-\lambda}$. Their wronskian $\text{Wr}(\lambda)$ is easily evaluated:

$$\text{Wr}(\lambda) = A_{\lambda+\eta} A_{-\lambda} - A_\lambda A_{-(\lambda+\eta)} = ([2\ell]!)^{-1} \left(\prod_{j=-\ell+1}^{\ell} \theta_1(2\lambda + 2j\eta) \right) W. \quad (2.16)$$

b) *The symmetry $x \leftrightarrow \lambda$.* Let $F(x)$ be an arbitrary function. Since $A_k(x, \lambda) = A_k(\lambda, x)$, it is clear from (2.12) that the result of action of the A_λ on the $F(x)$ is symmetric under the interchange of x and λ , i.e.,

$$A_\lambda(x, \partial_x) F(x) = A_x(\lambda, \partial_\lambda) F(\lambda), \quad (2.17)$$

where $A_\lambda(x, \partial_x)$ (respectively, $A_x(\lambda, \partial_\lambda)$) acts on the function of x (respectively, of λ).

c) *Even and odd difference operators.* Let Ξ be the operator changing the sign of x : $\Xi F(x) = F(-x)$. It is clear from the definition that $\Xi A_\lambda \Xi^{-1} = A_{-\lambda}$. We call difference operators O such that $\Xi O \Xi^{-1} = O$ (respectively, $\Xi O \Xi^{-1} = -O$) *even* (respectively, *odd*) operators. It can be proved [12] that for generic η any even difference operator commuting with L is a polynomial in L . In particular, $A_{k\eta} + A_{-k\eta}$ for $k \in \mathbf{Z}$ and $A_\lambda A_{-\lambda}$ for arbitrary $\lambda, \eta \in \mathbf{C}$ are polynomial functions of L .

d) *Relations between the transfer matrices $T_s(u)$ and the difference operators $A_{s\eta}$, $s \in \mathbf{Z}$.* From (2.13) we immediately conclude that $A_{-j\eta} = A_{j\eta}$, $-\ell \leq j \leq \ell$, so they are even operators. Then it follows from the above that the operators $A_{j\eta}$ with integer $-\ell \leq j \leq \ell$ are polynomial functions of \mathcal{S}_0 . So, similarly to (2.8), we define polynomials $A_{j\eta}(E)$ to be eigenvalues of the $A_{j\eta}$ on their common eigenfunction Ψ such that $\mathcal{S}_0 \Psi = E\Psi$. Comparing the fusion relation (2.5) for $s \leq 2\ell$ with (2.15), we identify

$$A_{(\ell-s)\eta} = \frac{[2\ell-s]!}{[2\ell]!} T_s(2\ell\eta - (s-1)\eta), \quad s = 0, 1, \dots, 2\ell, \quad (2.18)$$

whence (2.6) yields the determinant representation of the polynomials $A_{(\ell-s)\eta}(E)$:

$$A_{(\ell-s)\eta}(E) = \begin{bmatrix} \ell \\ s \end{bmatrix} \begin{bmatrix} 2\ell \\ s \end{bmatrix}^{-1} \det \left(E\delta_{i,j} + \frac{[-i]}{[\ell+1-i]} \delta_{i,j-1} + \frac{[2\ell+2-i]}{[\ell+1-i]} \delta_{i,j+1} \right)_{1 \leq i,j \leq s} \quad (2.19)$$

(here $0 \leq s \leq \ell$). The Baxter equation (2.15) gives the recurrence relation for these polynomials:

$$A_{(\ell-s-1)\eta}(E) = \frac{[\ell-s]}{[2\ell-s]} E A_{(\ell-s)\eta}(E) + \frac{[s]}{[2\ell-s]} A_{(\ell-s+1)\eta}(E) \quad (2.20)$$

with the initial conditions $A_{\ell\eta}(E) = 1$, $A_{(\ell-1)\eta}(E) = ([\ell]/[2\ell])E$. It is clear from (2.20) that

$$A_{(\ell-s)\eta}(-E) = (-1)^s A_{(\ell-s)\eta}(E), \quad 0 \leq s \leq \ell. \quad (2.21)$$

At last, we point out the relation

$$T_{2\ell+1}(0) = [2\ell]!(A_{(\ell+1)\eta} + A_{-(\ell+1)\eta}) \quad (2.22)$$

which follows e.g. from (2.11). The difference operator in the right hand side is even. Its eigenvalue is given by the polynomial $T_{2\ell+1}(0, E) = \lim_{u \rightarrow 0} T_{2\ell+1}(u, E)$ (see (2.9)).

3 Double-Bloch eigenfunctions and explicit form of the spectral curve

Let ℓ be a positive integer. For our current purpose it is more convenient to pass to the function

$$\psi(x) = \Psi(x) \left(\prod_{j=1}^{\ell} \theta_1(2x - 2j\eta) \right)^{-1}. \quad (3.1)$$

Then the eigenvalue equation for the L acquires the form

$$\psi(x + \eta) + \frac{\theta_1(2x + 2\ell\eta)\theta_1(2x - 2(\ell+1)\eta)}{\theta_1(2x)\theta_1(2x - 2\eta)} \psi(x - \eta) = E\psi(x) \quad (3.2)$$

which we also call the difference analogue of the Lamé equation. In this form, the coefficient function is *double-periodic* with periods $\frac{1}{2}$ and $\frac{\tau}{2}$, so it is natural to look for solutions in the class of *double-Bloch functions* [2], i.e., such that $\psi(x + \frac{1}{2}) = B_1\psi(x)$, $\psi(x + \frac{1}{2}\tau) = B_\tau\psi(x)$ with some constants B_1, B_τ .

Introduce the function

$$\Phi(x, \zeta) = \frac{\theta_1(\zeta + x)}{\theta_1(x)\theta_1(\zeta)}. \quad (3.3)$$

Its monodromy properties in x are $\Phi(x + 1, \zeta) = \Phi(x, \zeta)$, $\Phi(x + \tau, \zeta) = e^{-2\pi i \zeta} \Phi(x, \zeta)$, i.e., it is a double-Bloch function. The double-Bloch ansatz for the ψ is

$$\psi(x) = K^{x/\eta} \sum_{j=1}^{\ell} s_j(\zeta, K, E) \Phi(2x - 2j\eta, \zeta), \quad (3.4)$$

where ζ, K parametrize the Bloch multipliers of the function $\psi(x)$: $B_1 = K^{\frac{1}{2\eta}}$, $B_\tau = K^{\frac{\tau}{2\eta}} e^{-2\pi i \zeta}$. The coefficients s_j depend on the indicated parameters only.

Substituting (3.4) into (3.2) and computing the residues at the points $x = j\eta$, $j = 0, \dots, \ell$, we get $\ell + 1$ linear equations

$$\sum_{j=1}^{\ell} M_{ij} s_j = 0, \quad i = 0, 1, \dots, \ell, \quad (3.5)$$

for ℓ unknowns s_j . Matrix elements M_{ij} of this system are:

$$\begin{aligned} M_{ij} &= K\delta_{i,j-1} - E\delta_{i,j} + K^{-1} \frac{\theta_1(2(j+\ell+1)\eta)\theta_1(2(j-\ell)\eta)}{\theta_1(2(j+1)\eta)\theta_1(2j\eta)} \delta_{i,j+1} + \\ &+ K^{-1} \frac{\theta_1(\zeta - 2(j-i+1)\eta)}{\theta_1(\zeta)} \frac{\theta_1(2(i+\ell)\eta)\theta_1(2(i-\ell-1)\eta)}{\theta_1(2\eta)\theta_1(2(j-i+1)\eta)} (\delta_{i,0} - \delta_{i,1}). \end{aligned} \quad (3.6)$$

Here $i = 0, 1, \dots, \ell$, $j = 1, 2, \dots, \ell$. The overdetermined system (3.5) has nontrivial solutions if and only if rank of the rectangular matrix M_{ij} is less than ℓ . By $M^{(0)}$ and $M^{(1)}$ we denote $\ell \times \ell$ matrices obtained from M by deleting the rows with $i = 0$ and $i = 1$, respectively. Then the set of parameters ζ, K, E for which eq. (3.5) has solutions of the form (3.4) is determined by the system of two equations: $\det M^{(0)} = \det M^{(1)} = 0$. So the three parameters are constrained by two equations. They can be written out in a particularly compact form in terms of the family of polynomials (2.19) and the elliptic "binomial coefficients" (2.14). Expanding the determinants with respect to the first row and taking into account (2.19), (2.15), we come to the following statement.

Theorem 3.1 *The difference Lamé equation (3.2) has double-Bloch solutions of the form (3.4) if and only if the spectral parameters ζ, K, E obey the equations*

$$\begin{aligned} \sum_{j=0}^{\ell} (-1)^j K^{-j} \theta_1(\zeta - 2j\eta) \begin{bmatrix} \ell \\ j \end{bmatrix} A_{j\eta}(E) &= 0, \\ \sum_{j=0}^{\ell+1} (-1)^j K^{-j} \theta_1(\zeta - 2j\eta) \theta_1(2(j-1)\eta) \begin{bmatrix} \ell+1 \\ j \end{bmatrix} A_{(j-1)\eta}(E) &= 0, \end{aligned} \quad (3.7)$$

where $A_{j\eta}(E)$ are polynomials of $(\ell - |j|)$ -th degree explicitly given by (2.19). They coincide with eigenvalues of the commuting operators $A_{j\eta}$ introduced in (2.12) on their common eigenfunction Ψ such that $L\Psi = E\Psi$.

The equations (3.7) define a Riemann surface $\tilde{\Gamma}$, which covers the complex plane. The monodromy properties of the θ -function (see Appendix A) make it clear that this surface is invariant under the transformation

$$\zeta \mapsto \zeta + \tau, \quad K \mapsto K e^{4\pi i \eta}. \quad (3.8)$$

The factor of the $\tilde{\Gamma}$ over this transformation is an algebraic curve Γ , which is a ramified covering of the elliptic curve with periods $1, \tau$. It is clear from (3.7), (2.21) that the curve admits the involution

$$(\zeta, K, E) \mapsto (\zeta, -K, -E), \quad (3.9)$$

so the spectrum is symmetric with respect to the reflection $E \rightarrow -E$. Another result of [2], which is not so easy to see from (3.7), is that the curve Γ is at the same time a hyperelliptic curve.

Theorem 3.2 [2] *The curve Γ is a hyperelliptic curve of genus $g = 2\ell$. The hyperelliptic involution is given by*

$$(\zeta, K, E) \mapsto (4N\eta - \zeta, K^{-1}, E), \quad N = \frac{1}{2}\ell(\ell+1). \quad (3.10)$$

The points $P = (\zeta, K, E) \in \Gamma$ of the curve parametrize double-Bloch solutions $\psi(x) = \psi(x, P)$ to eq. (3.2), and the solution $\psi(x, P)$ corresponding to each point $P \in \Gamma$ is unique up to a constant multiplier.

To compare with [2], we note that the variables z, k used in that paper are related to ζ, K as follows: $\zeta = z + 2N\eta$, $K = k \left(\frac{\theta_1(z - 2\eta)}{\theta_1(z + 2\eta)} \right)^{\frac{1}{2}}$. In terms of z, k the coefficients in (3.7) become elliptic functions of z and the hyperelliptic involution is $z \rightarrow -z$, $k \rightarrow k^{-1}$.

The edges of bands $\pm E_i$ in (1.3) are values of the function $E = E(P)$ at the fixed points of the hyperelliptic involution. As is clear from (3.10), the fixed points lie above the points $\zeta = 2N\eta + \omega_a$, where ω_a are the half-periods: $\omega_1 = 0$, $\omega_2 = \frac{1}{2}$, $\omega_3 = \frac{1}{2}(1 + \tau)$, $\omega_4 = \frac{1}{2}\tau$. The corresponding values of K are determined from (3.8).

Corollary 3.1 *Let \mathcal{E}_a , $a = 1, \dots, 4$ be the set of common roots of the polynomial equations*

$$\begin{aligned} \sum_{j=0}^{\ell} \theta_a(2(N-j)\eta) \begin{bmatrix} \ell \\ j \end{bmatrix} A_{j\eta}(E) &= 0, \\ \sum_{j=0}^{\ell+1} \theta_a(2(N-j)\eta) \theta_1(2(j-1)\eta) \begin{bmatrix} \ell+1 \\ j \end{bmatrix} A_{(j-1)\eta}(E) &= 0, \end{aligned} \quad (3.11)$$

where $N = \frac{1}{2}\ell(\ell+1)$, and θ_a are Jacobi θ -functions. Then the set of the edges of bands $\pm E_i$ is the union of $\bigcup_{a=1}^4 \mathcal{E}_a$ and its image under the reflection $E \rightarrow -E$.

To obtain a more detailed information from equations (3.7), one can try to eliminate E and obtain a single equation connecting the two Bloch multipliers of the function (3.4) (parametrizing through ζ and K). However, this is not easy to do directly. A possible way out relies on the following simple lemma.

Lemma 3.1 *Let $\Psi(x)$ be any solution to the equation*

$$\frac{\theta_1(2x - 2\ell\eta)}{\theta_1(2x)} \Psi(x + \eta) + \frac{\theta_1(2x + 2\ell\eta)}{\theta_1(2x)} \Psi(x - \eta) = E\Psi(x) \quad (3.12)$$

in the class of entire functions on the complex plane of the variable x , then

$$\Psi(j\eta) = \Psi(-j\eta), \quad j = 1, 2, \dots, \ell. \quad (3.13)$$

This assertion follows from the specific form of the coefficients of eq. (3.12). Indeed, putting $x = 0$ in (3.12), we have $\Psi(\eta) = \Psi(-\eta)$. The proof can be completed by induction. At $x = \pm\ell\eta$ one of the coefficients in the l.h.s. of (3.12) vanishes, so the chain of relations (3.13) truncates at $j = \ell$. \blacksquare

Remark The conditions (3.13) resemble the "glueing conditions" for the Baker-Akhiezer function on rational curves with double points, where they are imposed on the Ψ with respect to its spectral parameter. However, contrary to that case, (3.13) is imposed in the x -plane.

Remarkably, the conditions (3.13) and the ansatz

$$\Psi(x) = K^{x/\eta} \left(\prod_{j=1}^{\ell} \theta_1(2x - 2j\eta) \right) \sum_{m=1}^{\ell} s_m(K, \zeta) \Phi(2x - 2m\eta, \zeta) \quad (3.14)$$

for Ψ (equivalent to the ansatz (3.4) for ψ) with the same function $\Phi(x, z)$ given by (3.3) allow one to find the relation between the Bloch multipliers even without explicit use of the difference Lamé equation (3.12). Plugging (3.14) into (3.13), we obtain ℓ equalities (for $m = 1, 2, \dots, \ell$):

$$K^m s_m = (-1)^\ell K^{-m} \theta_1(4m\eta) \left(\prod_{j=1, j \neq m}^{\ell} \frac{\theta_1(2(m+j)\eta)}{\theta_1(2(m-j)\eta)} \right) \sum_{n=1}^{\ell} \Phi(-2(m+n)\eta, \zeta) s_n. \quad (3.15)$$

This is a system of linear homogeneous equations for s_n . It has nontrivial solutions if and only if its determinant is equal to zero, whence we obtain the equation for ζ and K :

$$\det(K^{2m} \delta_{mn} + G_{mn}(\zeta))_{1 \leq m, n \leq \ell} = 0, \quad (3.16)$$

where

$$G_{mn}(\zeta) = (-1)^{\ell+1}[2m] \left(\prod_{j=1, j \neq m}^{\ell} \frac{[m+j]}{[m-j]} \right) \Phi(-2(m+n)\eta, \zeta).$$

This equation defines a curve Γ_e , which is the image of the spectral curve Γ under the projection $\Gamma \rightarrow \Gamma_e$ that takes (ζ, K, E) to (ζ, K) . A more explicit description of the curve Γ_e is given by the following proposition.

Proposition 3.1 *The equation of the spectral curve (3.16) can be represented in the form*

$$\sum_{j=0}^N (-1)^j C_j^{(\ell)}(\eta) \theta_1(\zeta - 4j\eta) K^{2(N-j)} = 0, \quad (3.17)$$

where $N = \frac{1}{2}\ell(\ell+1)$ and $C_j^{(\ell)}(\eta)$ are coefficients depending only on η (and τ) such that $C_j^{(\ell)}(\eta) = C_{N-j}^{(\ell)}(\eta)$, $C_0^{(\ell)}(\eta) = 1$.

Proof. The following direct proof allows us to find the explicit form of the coefficients $C_j^{(\ell)}$. To expand the determinant in powers of K , we make use of the identity

$$\det \left(\frac{\theta_1(x_i + x_j + \zeta)}{\theta_1(x_i + x_j)} \right)_{1 \leq i, j \leq n} = \frac{\theta_1^{n-1}(\zeta) \theta_1(\zeta + 2 \sum_{i=1}^n x_i)}{\prod_{i=1}^n \theta_1(2x_i)} \prod_{i < j} \frac{\theta_1^2(x_i - x_j)}{\theta_1^2(x_i + x_j)}$$

which is a particular case of the formula for the elliptic Cauchy determinant. Let Λ be the set $\{1, 2, \dots, \ell\}$. We use the following notation. For any subset $J \subseteq \Lambda$, $\Lambda \setminus J$ is its complement, $|J|$ is the number of its elements, and $\|J\| = \sum_{m \in J} m$. Setting $x_n = -2\eta n$, we have:

$$\begin{aligned} & \det(K^{2m} \delta_{mn} + G_{mn}(\zeta))_{1 \leq m, n \leq \ell} \\ &= \sum_{j=0}^N \frac{\theta_1(\zeta - 4j\eta)}{\theta_1(\zeta)} K^{2(N-j)} \sum_{J \subseteq \Lambda, \|J\|=j} (-1)^{\kappa(J)} \prod_{k \in J} \prod_{k' \in \Lambda \setminus J} \frac{\theta_1(2(k+k')\eta)}{\theta_1(2(k-k')\eta)}, \end{aligned} \quad (3.18)$$

where

$$\kappa(J) = |J|\ell + \frac{1}{2}|J|(|J|-1).$$

Thus, the coefficient $C_j^{(\ell)}$ reads

$$C_j^{(\ell)} = \sum_{J \subseteq \Lambda, \|J\|=j} (-1)^{\kappa(J)+j} \prod_{k \in J} \prod_{k' \in \Lambda \setminus J} \frac{\theta_1(2(k+k')\eta)}{\theta_1(2(k-k')\eta)}, \quad (3.19)$$

and the symmetry $j \leftrightarrow N - j$ is evident. \blacksquare

Note that the sum in (3.19) runs over partitions of the number j into *distinct* parts not exceeding ℓ . Examples are given in Appendix B.

The meaning of (3.17) is the same as is explained after equations (3.7): it defines a covering of the complex plane invariant under the map (3.8). This allows us to define the corresponding factor-curve which is precisely Γ_e . Therefore, Γ_e is a ramified covering of the elliptic curve with the modular parameter τ .

It is easily seen from (3.17) that Γ_e is invariant under the involution $(\zeta, K) \mapsto (4N\eta - \zeta, K^{-1})$. Eq. (3.12) allows one to express the function E through ζ, K by the following formula:

$$E = \frac{\theta_1(4\ell\eta)}{\theta_1(2\ell\eta)} \frac{\Psi((\ell-1)\eta)}{\Psi(\ell\eta)} = -\frac{s_{\ell-1}}{s_\ell} K^{-1} \frac{[1][2\ell]}{[\ell][\ell-1]}. \quad (3.20)$$

The coefficients $s_\ell, s_{\ell-1}$ are given by the corresponding minors of the matrix $K^{2m}\delta_{mn} + G_{mn}(\zeta)$. It can be shown that E is invariant under the above involution of Γ_e , so this involution coincides with (3.10). In terms of the Ψ -function, the hyperelliptic involution takes $\Psi(x)$ to $\Psi(-x)$. Note also that eq. (3.17) defines a singular curve. Indeed, the fixed points of the hyperelliptic involution are singular points of the curve Γ_e , i.e., both the ζ - and K -derivatives of the left hand side of (3.17) at these points equal to zero. In the neighbourhoods of these points different sheets of the curve intersect. The function E takes different values on these sheets (which are obtained by resolving the indeterminacy in (3.20)), so it resolves the singularities of the curve.

Remark The function $\Psi(x)$ is the common eigenfunction for all the commuting operators A_λ (2.12). Indeed, commutativity of L and A_λ implies that $\tilde{\Psi}_\lambda(x) = A_\lambda\Psi(x)$ is an eigenfunction of L with the same eigenvalue E . By Theorem 3.2, $\tilde{\Psi}$ is proportional to Ψ : $\tilde{\Psi}_\lambda(x) = g(\lambda)\Psi(x)$. From the symmetry (2.17) and the normalization condition $A_{\ell\eta} = 1$ we have

$$A_\lambda\Psi(x) = \frac{\Psi(\lambda)}{\Psi(\ell\eta)}\Psi(x).$$

Let us conclude this section by examining the behaviour of the spectral curve in the vicinity of its "infinite points", i.e., the points at which the function E has poles. From either (3.7) or (3.17), (3.20) we conclude that there are two such points: $\infty_+ = (\zeta \rightarrow 0, K \rightarrow \infty, E \rightarrow \infty)$ and $\infty_- = (\zeta \rightarrow 4N\eta, K \rightarrow 0, E \rightarrow \infty)$. In the neighbourhood of ∞_\pm we have $E = K^{\pm 1} + o(K^{\pm 1})$, while the leading terms of the function K are:

$$K^2 = -\frac{1}{\theta_1(\zeta)} \frac{[\ell][\ell+1]}{[1]} + O(1), \quad \zeta \rightarrow 0,$$

$$K^2 = \theta_1(\zeta - 4N\eta) \frac{[1]}{[\ell][\ell+1]} + o(\zeta - 4N\eta), \quad \zeta \rightarrow 4N\eta.$$

The Baker-Akhiezer function

$$\Psi_{BA}(x, P) = \frac{\Psi(x, P)}{\Psi(\ell\eta, P)}, \quad P = (\zeta, K, E) \in \Gamma,$$

is easily seen to have the following asymptotics as $P \rightarrow \infty_{\pm}$:

$$\Psi_{BA}(x, P) = K^{\frac{x}{\eta} \pm \ell} \left(\xi_0^{\pm}(x) + O(K^{\mp 1}) \right) \quad (3.21)$$

with some functions $\xi_0^{\pm}(x)$, i.e., the pole divisor of the Baker-Akhiezer function is concentrated at ∞_{\pm} .

4 Explicit hyperelliptic realizations

In this section we obtain the equation of the spectral curve of the difference Lamé operator L , which has the explicit hyperelliptic form. This equation contains two variables: E and z . The latter is the eigenvalue of the operator $A_{(\ell+1)\eta} \equiv A$, which commutes with L , on their common eigenfunction Ψ : $L\Psi = E\Psi$, $A\Psi = z\Psi$. Recall that the operator $\bar{A} = A_{-(\ell+1)\eta}$ commutes with both A and L . Let us write out the trivial identity $A^2 - (A + \bar{A})A + A\bar{A} = 0$ and act by both sides on the Ψ . Taking into account (2.22), we get $z^2 - ([2\ell]!)^{-1}T_{2\ell+1}(0, E) + D_{2\ell}(E) = 0$, or

$$z + \frac{D_{2\ell}(E)}{z} = ([2\ell]!)^{-1}T_{2\ell+1}(0, E), \quad (4.1)$$

where $D_{2\ell}(E)$ is a polynomial of E of degree 2ℓ and $T_{2\ell+1}(0, E) = \lim_{u \rightarrow 0} T_{2\ell+1}(u, E)$ is the polynomial of E of degree $2\ell + 1$. They enjoy the properties $D_{2\ell}(-E) = D_{2\ell}(E)$, $T_{2\ell+1}(0, -E) = -T_{2\ell+1}(0, E)$. Recall that

$$L = c_-(x)e^{\eta\partial_x} + c_+(x)e^{-\eta\partial_x}, \quad A = \sum_{k=0}^{\ell} a_{2k+1}(x)e^{(2k+1)\eta\partial_x}, \quad (4.2)$$

where the coefficient functions are:

$$c_{\pm}(x) = \frac{\theta_1(2x \pm \ell\eta)}{\theta_1(2x)},$$

$$a_{2k+1}(x) = (-1)^k \frac{\begin{bmatrix} 2\ell+1 \\ \ell-k \end{bmatrix}}{\begin{bmatrix} 2\ell \\ \ell \end{bmatrix}} \prod_{j=0}^{\ell-k-1} \frac{\theta_1(2x + 2(\ell-j)\eta)}{\theta_1(2x + 2(\ell+k-j+1)\eta)} \prod_{j=0}^{k-1} \frac{\theta_1(2x - 2(\ell-j)\eta)}{\theta_1(2x + 2(k+j+1)\eta)} \quad (4.3)$$

(see (2.12), (2.13)).

Introducing $w = 2z - ([2\ell]!)^{-1}T_{2\ell+1}(0, E)$, we rewrite (4.1) in the customary hyperelliptic form:

$$w^2 = ([2\ell]!)^{-2}(T_{2\ell+1}(0, E))^2 - 4D_{2\ell}(E) = \left[\frac{2\ell}{\ell} \right]^{-2} P_{2\ell+1}(E^2), \quad (4.4)$$

where $P_{2\ell+1}(E^2) = \prod_{i=1}^{2\ell+1}(E^2 - E_i^2)$. Note that w is the eigenvalue of the operator $W = A - \bar{A}$.

To find out the explicit form of $D_{2\ell}(E)$, we make use of the following simple argument: given a commuting pair $[L, A] = 0$ of operators of finite order, the spectral problem for L is reduced to an eigenvalue problem for a finite matrix with elements depending on the eigenvalue of A (the "spectral parameter"). Specifically, let $\Psi(x)$ be their common eigenfunction. We set

$$\Psi_j = \Psi_j(x) = \Psi(x + j\eta). \quad (4.5)$$

The equation $A\Psi = z\Psi$ allows us to express Ψ_0 and $\Psi_{2\ell+2}$ through $\Psi_1, \Psi_2, \dots, \Psi_{2\ell+1}$:

$$\begin{aligned} \Psi_0 &= z^{-1} \sum_{k=0}^{\ell} a_{2k+1}(x) \Psi_{2k+1}, \\ \Psi_{2\ell+2} &= za_{2\ell+1}^{-1}(x + \eta) \Psi_1 - \sum_{k=1}^{\ell} \frac{a_{2k-1}(x + \eta)}{a_{2\ell+1}(x + \eta)} \Psi_{2k}. \end{aligned} \quad (4.6)$$

Now the spectral problem $L\Psi = E\Psi$ can be rewritten as a homogeneous linear system for $\Psi_1, \dots, \Psi_{2\ell+1}$. Equating its determinant to zero, we get a relation between z and E , which is the equation of our spectral curve. Its independence of the value of x follows, eventually, from commutativity of L and A .

To be more precise, introduce the vector-function $\vec{\Psi}(x)$ with components $\Psi_j(x)$ (see (4.5)), $j = 1, \dots, 2\ell + 1$. The eigenvalue equation for A can be rewritten in the form

$$\vec{\Psi}(x - \eta) = \mathbf{A}(x, z)\vec{\Psi}(x), \quad (4.7)$$

where the matrix $\mathbf{A}(x, z)$ reads

$$\mathbf{A}(x, z) = \begin{pmatrix} z^{-1}a_1(x) & 0 & z^{-1}a_3(x) & \dots & 0 & z^{-1}a_{2\ell+1}(x) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (4.8)$$

The scalar equation $L\Psi = E\Psi$ is then rewritten in the matrix form $\mathbf{L}(x, z)\vec{\Psi}(x) = E\vec{\Psi}(x)$, where

$$\mathbf{L}(x, z) = \mathbf{C}_+(x)\mathbf{A}(x, z) + \mathbf{C}_-(x)\mathbf{A}^{-1}(x + \eta, z), \quad (4.9)$$

$$\mathbf{C}_{\pm} = \text{diag} \left\{ c_{\pm}(x + \eta), c_{\pm}(x + 2\eta), \dots, c_{\pm}(x + (2\ell + 1)\eta) \right\}.$$

So, we have assigned to the scalar operators A and L the $(2\ell + 1) \times (2\ell + 1)$ -matrices \mathbf{A} and \mathbf{L} respectively.

Lemma 4.1 *The commutativity condition $[L, A] = 0$ is equivalent to the Lax equation*

$$\mathbf{L}(x - \eta, z)\mathbf{A}(x, z) = \mathbf{A}(x, z)\mathbf{L}(x, z) \quad (4.10)$$

for arbitrary values of z .

Proof. This equality holds identically for the rows from the second to the last one. The first row gives the set of relations for coefficients of the L, A , which are equivalent to their commutativity. ■

The system $\mathbf{L}(x, z)\vec{\Psi}(x) = E\vec{\Psi}(x)$ has nontrivial solutions if and only if E and z are connected by the equation of the spectral curve:

$$\det(\mathbf{L}_{ij}(x, z) - E\delta_{ij})_{1 \leq i, j \leq 2\ell+1} = 0. \quad (4.11)$$

It is easy to see that this determinant does not depend on x . Indeed, denote the left hand side of (4.11) by $f(x)$. It follows from (4.9) that $\mathbf{L}_{mn}(x + \frac{1}{2}, z) = \mathbf{L}_{mn}(x, z)$, $\mathbf{L}_{mn}(x + \frac{1}{2}\tau, z) = e^{-4\pi i\ell(m-n)\eta}\mathbf{L}_{mn}(x, z)$, whence $f(x)$ is a double-periodic function of the variable $y = 2x$ with periods 1 and τ and with finite number of possible poles. At the same time, (4.10) implies that $f(x)$ has one and the same value at infinite number of points $x + m\eta$, $m \in \mathbf{Z}$. Thus, $f(x) = \text{const}$.

Extracting the z -dependence of the determinant in (4.11), we can write

$$\det(\mathbf{L}_{ij}(x, z) - E\delta_{ij}) = z(-1)^\ell \begin{bmatrix} 2\ell \\ \ell \end{bmatrix} + F + z^{-1}G,$$

where F and G are polynomials of E . According to the above argument, they do not depend on x . It is convenient to evaluate F at $x \rightarrow -(\ell + 1)\eta$. This should be done with some care because some matrix elements are singular at this point. Using (2.9), we find:

$$\begin{aligned} F &= \left(\prod_{k=1, k \neq \ell+1}^{2\ell+1} \theta_1(2x + 2k\eta) \right)^{-1} T_{2\ell+1}(2x + 2(\ell+1)\eta, -E) \\ &\quad + \text{two "unwanted" determinants.} \end{aligned}$$

At $x \rightarrow -(\ell + 1)\eta$ the first term yields $(-1)^{\ell+1}([\ell]!)^{-2}T_{2\ell+1}(0, E)$, and each of the two "unwanted" terms tends to zero. The simplest determinant representation for G is obtained at $x = \ell\eta$ when almost all elements of the first row are equal to zero. Skipping the details, we present the result.

The equation of the curve has the form (4.1), where

$$D_{2\ell}(E) = (-1)^\ell \frac{[2\ell + 1]}{[\ell + 1]} \begin{bmatrix} 2\ell \\ \ell \end{bmatrix}^{-1} \det(\mathbf{D}_{ij})_{1 \leq i, j \leq 2\ell}. \quad (4.12)$$

The $(2\ell \times 2\ell)$ -matrix \mathbf{D}_{ij} reads

$$\mathbf{D}_{ij} = \begin{pmatrix} -E & \frac{[2]}{[\ell+2]} & 0 & 0 & \dots & 0 & 0 & 0 \\ \frac{[2\ell+3]}{[\ell+3]} & -E & \frac{[3]}{[\ell+3]} & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{[2\ell+4]}{[\ell+4]} & -E & \frac{[4]}{[\ell+4]} & \dots & 0 & 0 & 0 \\ \dots & \dots \\ \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{[4\ell]}{[3\ell]} & -E & \frac{[2\ell]}{[3\ell]} \\ d_2 & 0 & d_4 & 0 & \dots & 0 & d_{2\ell} + \frac{[4\ell+1]}{[3\ell+1]} & -E \end{pmatrix}, \quad (4.13)$$

where the coefficients d_{2k} entering the last line are:

$$d_{2k} = (-1)^{\ell-k} \frac{[\ell+2k]}{[k]} \begin{bmatrix} 3\ell \\ \ell \end{bmatrix} \begin{bmatrix} 2\ell+1 \\ \ell+k \end{bmatrix} \begin{bmatrix} 2\ell+k+1 \\ k \end{bmatrix}^{-1}, \quad k = 1, \dots, \ell.$$

It is not difficult to see that the element $\mathbf{D}_{2\ell,2\ell-1}$ is equal to

$$d_{2\ell} + \frac{[4\ell+1]}{[3\ell+1]} = \frac{[\ell+1][2\ell]^2}{[1][\ell]^2}.$$

The polynomials $D_{2\ell}(E)$ for small values of ℓ are written out in Appendix B.

The size of the determinant in the equation defining the curve can be reduced for the price of more complicated matrix elements. Let us mention two examples.

The first one is 2×2 determinant representation, which is in certain sense "dual" to (4.10). The duality means that the roles of \mathbf{A} - and \mathbf{L} -matrices are interchanged: according to the same scheme, the spectral problem for A is reduced to an eigenvalue problem for a 2×2 -matrix with elements depending on the eigenvalue E of L . Specifically, set

$$\mathcal{L}(x, E) = \begin{pmatrix} 0 & 1 \\ -\frac{c_+(x+\eta)}{c_-(x+\eta)} & Ec_-^{-1}(x+\eta) \end{pmatrix} \quad (4.14)$$

and rewrite the equation $L\Psi = E\Psi$ in the form $\begin{pmatrix} \Psi(x+\eta) \\ \Psi(x+2\eta) \end{pmatrix} = \mathcal{L}(x, E) \begin{pmatrix} \Psi(x) \\ \Psi(x+\eta) \end{pmatrix}$. Introduce the "monodromy matrix"

$$\mathcal{A}(x, E) = \sum_{k=0}^{\ell} \begin{pmatrix} a_{2k+1}(x) & 0 \\ 0 & a_{2k+1}(x+\eta) \end{pmatrix} \overleftarrow{\prod}_{2k \geq j \geq 0} \mathcal{L}(x+j\eta),$$

where the arrow indicates the ordered product of matrices, and $a_{2k+1}(x)$ are the same as in (4.2). Then the spectral problem $A\Psi = z\Psi$ is rewritten as $\mathcal{A}(x, E) \begin{pmatrix} \Psi(x) \\ \Psi(x+\eta) \end{pmatrix} =$

$z \begin{pmatrix} \Psi(x) \\ \Psi(x + \eta) \end{pmatrix}$, and the Lax equation

$$\mathcal{A}(x + \eta, E)\mathcal{L}(x, E) = \mathcal{L}(x, E)\mathcal{A}(x, E) \quad (4.15)$$

holds true provided the operators L and A commute. Note that now $\mathcal{A}(x, E)$ plays the role of the Lax matrix (cf. (4.10)). The equation of the spectral curve reads

$$\det(\mathcal{A}(x, E) - z) = 0. \quad (4.16)$$

Due to the Lax equation (4.15) it does not depend on x .

The second one is $\ell \times \ell$ determinant representation for the factor-curve obtained by factorization over the reflection $E \rightarrow -E$. In the operator language, this amounts to finding a polynomial relation between the commuting operators $A_{(\ell-2)\eta}$ and $A_{(\ell+2)\eta}$. As it follows from (2.15), their eigenvalues (ε and ξ , respectively) are connected with E, z by the formulas

$$\varepsilon = \frac{[\ell-1][\ell]}{[2\ell][2\ell-1]} \left(E^2 + \frac{[1][2\ell]}{[\ell-1][\ell]} \right), \quad \xi = \frac{[\ell+1]}{[1]} \left(Ez - \frac{[2\ell+1]}{[\ell+1]} \right).$$

The equation of the curve can be derived in the same way as (4.11).

5 Remarks on representations of the Sklyanin algebra

In this section we make a few remarks relating the above material to representation theory of the Sklyanin algebra.

Let $\mathcal{S}_a^{(\ell)}$ denote the generators of the Sklyanin algebra realized by difference operators as in (2.2). To make connections between the commuting family A_λ (2.12) and representations of the Sklyanin algebra explicit, we begin with a simple reformulation of the Novikov equation $[L, V] = 0$ for coefficients of an operator V commuting with $L = \mathcal{S}_0^{(\ell)}$. Suppose $v(x, y)$ is any solution to the equation

$$\nabla v(x, y) = 0, \quad (5.1)$$

where $\nabla = \mathcal{S}_0^{(\ell)}(x, \partial_x) - \mathcal{S}_0^{(\ell)}(y, \partial_y)$. (The operator $\mathcal{S}_0^{(\ell)}(x, \partial_x)$ acts to the variable x , etc.) Then the operator

$$V = \sum V_j(x) e^{j\eta\partial_x} \quad (5.2)$$

with the coefficients

$$V_j(x) = v(x, x + j\eta) \left(\prod_{k=1}^{\ell} \theta_1(2x + 2(j-k)\eta) \theta_1(2x + 2(j+k)\eta) \right)^{-1}$$

commutes with $\mathcal{S}_0^{(\ell)}$. In general, the operators V given by this construction are of infinite order, i.e. the sum in (5.2) is infinite. For the family of commuting operators A_λ (2.12) the sum is *finite*. This corresponds to some very particular solutions to eq. (5.1).

Recall that the Sklyanin algebra can be realized [20] by certain difference operators in two variables acting on the invariant subspace of solutions to eq. (5.1). Taking into

account the isomorphism between difference operators commuting with the operator $\mathcal{S}_0^{(\ell)}$ and meromorphic functions on its spectral curve Γ with poles only at ∞_{\pm} , we conclude that the Sklyanin algebra acts in the space of such functions on Γ . It would be very interesting to find the explicit form of this action.

In the rest of this section we present some results on the role of the commuting operators A_λ in representations of the Sklyanin algebra. First, we show that they have the same invariant subspace as the generators of the Sklyanin algebra (2.2). Second, the operator $W = A - \bar{A}$ is shown to "intertwine" representations of spins ℓ and $-\ell - 1$.

Proposition 5.1 *Let ℓ be a positive integer. Then the operators A_λ (2.12) preserve the space $\Theta_{4\ell}^+$.*

Sketch of proof. Let $F(x) \in \Theta_{4\ell}^+$. Then the monodromy properties of $\tilde{F}(x) = (A_\lambda F)(x)$ are the same as those of θ -functions of order 4ℓ (this is easily seen from (2.13)). Next, a further inspection of (2.12), (2.13) shows that the condition $F(x) = F(-x)$ is enough for cancellation of all poles of $\tilde{F}(x)$. Therefore, $\tilde{F}(x) \in \Theta_{4\ell}$. It remains to prove that $\tilde{F}(x)$ actually belongs to $\Theta_{4\ell}^+$. Set $f(x) = \tilde{F}(x) - \tilde{F}(-x)$, then $f(x) = A_x(\lambda, \partial_\lambda)F(\lambda) - A_{-x}(\lambda, \partial_\lambda)F(\lambda)$, where the $x \leftrightarrow \lambda$ symmetry (2.17) is used. Since $f(x) \in \Theta_{4\ell}$, it is enough to verify the equality $f(x) = 0$ in 4ℓ points $x = m\eta + \omega_a$, $m = 1, \dots, \ell$, where ω_a are the half-periods. This is easy to do if to recall the operator identity $A_{m\eta} = A_{-m\eta}$ for $m = 1, \dots, \ell$. \blacksquare

Corollary 5.1 *For any $\lambda \in \mathbf{C}$ the operator $A_\lambda - A_{-\lambda}$ annihilates the space $\Theta_{4\ell}^+$.*

Indeed, for $F(x) \in \Theta_{4\ell}^+$ the function $(A_\lambda - A_{-\lambda})F(x)$ is simultaneously odd and even.

Remark Recall the involution (3.9) that changes the sign of E . This involution takes the eigenfunction $\Psi(x)$ (see 3.14) to $e^{\frac{i\pi x}{\eta}}\Psi(x)$. Clearly, the operators A_λ preserve the space $e^{\frac{i\pi x}{\eta}}\Theta_{4\ell}^+$ as well, so $A_\lambda - A_{-\lambda}$ annihilates the space $\Theta_{4\ell}^+ \oplus e^{\frac{i\pi x}{\eta}}\Theta_{4\ell}^+$ spanned by eigenfunctions of the difference Lamé operator at the edges of bands (cf. [2]).

To formulate the next proposition, it is convenient to slightly modify the operator $W = A - \bar{A}$. Let us introduce the operator

$$\check{W} = (-1)^\ell \begin{bmatrix} 2\ell \\ \ell \end{bmatrix} \varphi_\ell^{-1}(x) W, \quad (5.3)$$

where $\varphi_\ell(x) = \prod_{j=0}^{2\ell} \theta_1(2x + 2(j - \ell)\eta)$. The explicit formula for \check{W} can be written in the form that has sense not only for $\ell \in \mathbf{Z}_+$ but also for $\ell \in \mathbf{Z}_+ + \frac{1}{2}$:

$$\check{W} = \sum_{k=0}^{2\ell+1} (-1)^k \begin{bmatrix} 2\ell+1 \\ k \end{bmatrix} \frac{\theta_1(2x + 2(2\ell - 2k + 1)\eta)}{\prod_{j=0}^{2\ell-k+1} \theta_1(2x + 2j\eta) \prod_{j'=1}^k \theta_1(2x - 2j'\eta)} e^{(2\ell-2k+1)\eta\partial_x}. \quad (5.4)$$

Here and below the dependence of the \check{W} on ℓ is not indicated explicitly. The following proposition is proved by a straightforward verification using some identities for the θ -functions.

Proposition 5.2 For $\ell \in \frac{1}{2}\mathbf{Z}_+$, the operator \check{W} "intertwines" representations of spin ℓ and of spin $-(\ell+1)$:

$$\mathcal{S}_a^{(-\ell-1)}\check{W} = \check{W}\mathcal{S}_a^{(\ell)}, \quad a = 0, \dots, 3. \quad (5.5)$$

The same intertwining relation can be written for the quantum L-operator (2.1): $L^{(-\ell-1)}\check{W} = \check{W}L^{(\ell)}$.

Remark In case of the algebra sl_2 the intertwining operator between representations of spins ℓ and $-\ell-1$ (realized by differential operators in x) is $(d/dx)^{2\ell+1}$. It annihilates the linear space of polynomials of degree $\leq 2\ell$.

Note that the operator \check{W} is not invertible. By Corollary 5.1, for $\ell \in \mathbf{Z}_+$ \check{W} annihilates the space $\Theta_{4\ell}^+ \oplus e^{\frac{i\pi x}{\eta}}\Theta_{4\ell}^+$ (see the remark after Corollary 5.1). As is mentioned above, this is precisely the space spanned by eigenfunctions of the difference Lamé operator at the edges of bands. So, in this way we obtain another proof of the result of [2]: the eigenfunctions of L at the edges of bands span a $(4\ell+2)$ -dimensional functional subspace, which is invariant for all Sklyanin's operators \mathcal{S}_a . The corresponding $(4\ell+2)$ -dimensional representation of the Sklyanin algebra is the direct sum of two equivalent $(2\ell+1)$ -dimensional irreducible representations.

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Appendix A. Theta-functions

We use the following definition of the Jacobi θ -functions:

$$\begin{aligned} \theta_1(x|\tau) &= \sum_{k \in \mathbf{Z}} \exp \left(\pi i \tau \left(k + \frac{1}{2} \right)^2 + 2\pi i \left(x + \frac{1}{2} \right) \left(k + \frac{1}{2} \right) \right), \\ \theta_2(x|\tau) &= \sum_{k \in \mathbf{Z}} \exp \left(\pi i \tau \left(k + \frac{1}{2} \right)^2 + 2\pi i x \left(k + \frac{1}{2} \right) \right), \\ \theta_3(x|\tau) &= \sum_{k \in \mathbf{Z}} \exp \left(\pi i \tau k^2 + 2\pi i x k \right), \\ \theta_4(x|\tau) &= \sum_{k \in \mathbf{Z}} \exp \left(\pi i \tau k^2 + 2\pi i \left(x + \frac{1}{2} \right) k \right). \end{aligned} \quad (\text{A1})$$

Throughout the paper we write $\theta_a(x|\tau) = \theta_a(x)$. The frequently used transformation properties for shifts by (half) periods are:

$$\theta_a(x \pm 1) = (-1)^{\delta_{a,1} + \delta_{a,2}} \theta_a(x), \quad \theta_a(x \pm \tau) = (-1)^{\delta_{a,1} + \delta_{a,4}} e^{-\pi i \tau \mp 2\pi i x} \theta_a(x), \quad (\text{A2})$$

$$\begin{aligned}
\theta_1(x \pm \frac{1}{2}) &= \pm \theta_2(x), \\
\theta_1(x \pm \frac{\tau}{2}) &= \pm i e^{-\frac{1}{4}\pi i \tau \mp \pi i x} \theta_4(x), \\
\theta_1(x \pm \frac{1+\tau}{2}) &= \pm e^{-\frac{1}{4}\pi i \tau \mp \pi i x} \theta_3(x),
\end{aligned} \tag{A3}$$

By Θ_n we denote the space of θ -functions of order n , i.e., entire functions $F(x)$, $x \in \mathbf{C}$, such that

$$F(x+1) = F(x), \quad F(x+\tau) = (-1)^n e^{-\pi i n \tau - 2\pi i n x} F(x). \tag{A4}$$

It is easy to see that $\dim \Theta_n = n$. Let $F(x) \in \Theta_n$, then $F(x)$ has a multiplicative representation of the form

$$F(x) = c \prod_{i=1}^n \theta_1(x - x_i), \quad \sum_{i=1}^n x_i = 0,$$

where c is a constant. Imposing, in addition to (A4), the condition $F(-x) = F(x)$, we define the space $\Theta_n^+ \subset \Theta_n$ of even θ -functions of order n , which plays the important role in representations of the Sklyanin algebra. If n is an even number, then $\dim \Theta_n^+ = \frac{1}{2}n + 1$.

Appendix B

Here we explicitly write out equation (3.17) for small values of ℓ . The "elliptic number" notation $[n] \equiv \theta_1(2n\eta)$ is used.

$$\begin{aligned}
\ell = 1 : \quad & \theta_1(\zeta)K^2 - \theta_1(\zeta - 4\eta) = 0, \\
\ell = 2 : \quad & \theta_1(\zeta)K^6 - \frac{[3]}{[1]}\theta_1(\zeta - 4\eta)K^4 + \frac{[3]}{[1]}\theta_1(\zeta - 8\eta)K^2 - \theta_1(\zeta - 12\eta) = 0, \\
\ell = 3 : \quad & \theta_1(\zeta)K^{12} - \frac{[3][4]}{[1][2]}\theta_1(\zeta - 4\eta)K^{10} + \frac{[3][5]}{[1]^2}\theta_1(\zeta - 8\eta)K^8 - 2\frac{[4][5]}{[1][2]}\theta_1(\zeta - 12\eta)K^6 + \\
& + \frac{[3][5]}{[1]^2}\theta_1(\zeta - 16\eta)K^4 - \frac{[3][4]}{[1][2]}\theta_1(\zeta - 20\eta)K^2 + \theta_1(\zeta - 24\eta) = 0, \\
\ell = 4 : \quad & \theta_1(\zeta)K^{20} - \frac{[4][5]}{[1][2]}\theta_1(\zeta - 4\eta)K^{18} + \frac{[3][5][6]}{[1]^2[2]}\theta_1(\zeta - 8\eta)K^{16} - \\
& - \left(\frac{[4][5][7]}{[1]^2[2]} + \frac{[4][5]^2[6]}{[1][2]^2[3]} \right) \theta_1(\zeta - 12\eta)K^{14} + \left(\frac{[5][6][7]}{[1][2][3]} + \frac{[5]^2[7]}{[1]^3} \right) \theta_1(\zeta - 16\eta)K^{12} - \\
& - 2\frac{[3][4][6][7]}{[1]^2[2]^2}\theta_1(\zeta - 20\eta)K^{10} + \\
& + \left(\frac{[5][6][7]}{[1][2][3]} + \frac{[5]^2[7]}{[1]^3} \right) \theta_1(\zeta - 24\eta)K^8 - \left(\frac{[4][5][7]}{[1]^2[2]} + \frac{[4][5]^2[6]}{[1][2]^2[3]} \right) \theta_1(\zeta - 28\eta)K^6 + \\
& + \frac{[3][5][6]}{[1]^2[2]}\theta_1(\zeta - 32\eta)K^4 - \frac{[4][5]}{[1][2]}\theta_1(\zeta - 36\eta)K^2 + \theta_1(\zeta - 40\eta) = 0.
\end{aligned}$$

Let us present the explicit form of the polynomials $T_{2\ell+1}(0, E)$ and $D_{2\ell}(E)$ (see (4.1)) for $\ell = 1, 2$.

$$T_3(0, E) = -[1]^2 \left\{ E^3 - \left(\frac{[1][4]}{[2][3]} + \frac{[2]^4}{[1]^3[3]} \right) E \right\},$$

$$T_5(0, E) = [1]^2[2]^2 \left\{ E^5 + \left(3 \frac{[4]}{[2]} - \frac{[2]^4}{[1]^4} \right) E^3 + \left(\frac{[4]^2}{[2]^2} + \frac{[3]^3}{[1]^3} - \frac{[1][6]}{[2][3]} \right) E \right\},$$

$$D_2(E) = -\frac{[1]}{[2]} \left(\frac{[3]}{[2]} E^2 - \frac{[2]^3}{[1]^3} \right),$$

$$D_4(E) = \frac{[1]}{[3]} \left\{ \frac{[2][5]}{[3][4]} E^4 - \left(\frac{[2]^2[7]}{[3][4]^2} + \frac{[4]^2[5]}{[1][2][6]} + \frac{[2][8]}{[4][6]} \right) E^2 + \frac{[4][5]}{[1][6]} \left(\frac{[7]}{[5]} + \frac{[5]}{[1]} \right) \right\}.$$

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